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On The Zeros of Polar Derivatives

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Abstract: **We extend some existing results on the zeros of polar derivatives of polynomials by considering more general coefficient conditions. As special cases the extended results yield much simpler expressions for the upper bounds of zeros than those of the existing results.**
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Keywords: **Zeros of polynomial, Eneström-Kakeya theorem, Polar derivatives.**

1. INTRODUCTION

Let denote $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ is a polar derivative of P(z) with respect to real number α. The polynomial $D_\alpha P(z)$ is of degree at most n-1 and it generalizes the ordinary derivative in the sense that $\lim_{\alpha\to\infty} \frac{D_\alpha P(z)}{\alpha} = P'(z)$. Many results on the location of zeros of polynomials are available in the literature. In literature [3-5] attempts have been made to extend and generalize the Enestrom-Kakeya theorem. Existing results in the literature also show that there is a need to find bounds for special polynomials, for example, for those having restrictions on the coefficient, there is always need for refinement of results in this subject to find location of zeros of polar derivatives of polynomials. Among them the Eneström-Kakeya theorem [1-2] given below is well known in the theory of zero distribution of polynomials.

Theorem (Enestrom-Kakeya theorem): Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n* such that $0 < a_0 \le a_1 \le a_2 \le \cdots \le a_n$ then all the zeros of P(z) lie in |z|≤1.

Here we establish zeros of polar derivatives of polynomial by using Eneström-Kakeya Theorem.

Theorem 1. Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n and let denote* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *is a polar derivative of P(z) with respect to real number α* such that

$na_0 \le (n-1)a_1 \le (n-2)a_2 \le \cdots \le 3a_{n-3} \le 2a_{n-2} \le a_{n-1}$ *if* $\alpha = 0$ then all the zeros of polar derivative $D_0 P(z)$ lie in $|z| \le \frac{1}{|a_{n-1}|}[\, a_{n-1} - na_0 + |na_0|\,]$.

Corollary 1. Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n and let denote* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *is a polar derivative of P(z) with respect to real number α* such that

$0 < na_0 \le (n-1)a_1 \le (n-2)a_2 \le \cdots \le 3a_{n-3} \le 2a_{n-2} \le a_{n-1}$ *if* $\alpha = 0$ then all the zeros of polar derivative $D_0 P(z)$ lie in $|z| \le 1$.

Remark 1.

By taking $a_i > 0$ *for* $i = 0,1,2,\dots,n-1$, in theorem 1, then it reduces to Corollary 2.

Theorem 2. Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n and let denote* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *is a polar derivative of P(z) with respect to real number α* such that

$na_0 \ge (n-1)a_1 \ge (n-2)a_2 \ge \cdots \ge 3a_{n-3} \ge 2a_{n-2} \ge a_{n-1}$ *if* $\alpha = 0$ then all the zeros of polar derivative $D_0 P(z)$ lie in $|z| \le \frac{1}{|a_{n-1}|}[|na_0| + na_0 - a_{n-1}\,]$.

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Corollary 2. Let P(z)= $\sum_{i=0}^{n} a_i z^i$ *be a polynomial of degree n and let denote* $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ *is a polar derivative of P(z) with respect to real number* α such that

$na_0 \geq (n-1)a_1 \geq (n-2)a_2 \geq \cdots \geq 3a_{n-3} \geq 2a_{n-2} \geq a_{n-1} > 0$ $if\ \alpha = 0$ then all the zeros of polar derivative $D_0P(z)$ lie in $|z| \leq \frac{1}{a_{n-1}}[2na_0 - a_{n-1}]$.

Remark 2.

By taking $a_i > 0\ for\ i = 0,1,2,\ldots,n-1$, in theorem 2, then it reduces to Corollary 2.

2. PROOFS OF THE THEOREMS

Proof of the Theorem 1.

Let P(z) = $a_n z^n + a_{n-1}z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$ be a polynomial of degree n

Let denote $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ is polar derivative of P(z) with respect to real number α of degree n-1.

If $\alpha = 0$ then $D_0P(z) = nP(z) - zP'(z)$

$\Rightarrow D_0P(z) = a_{n-1}z^{n-1} + 2a_{n-2}z^{n-2} + 3a_{n-3}z^{n-3} + \cdots + (n-2)a_2z^2 + (n-1)a_1z + na_0$

Let us consider the polynomial Q(z) = $(1 - z)\, D_0P(z)$ so that

Q(z)=(1-z)($a_{n-1}z^{n-1} + 2a_{n-2}z^{n-2} + 3a_{n-3}z^{n-3} + \cdots + (n-2)a_2z^2 + (n-1)a_1z + na_0$)

$$= -a_{n-1}z^n + (a_{n-1} - 2a_{n-2})z^{n-1} + (2a_{n-2} - 3a_{n-3})z^{n-2} + \cdots + [(n-3)a_3 - (n-2)a_2]z^3 + [(n-2)a_2 - (n-1)a_1]z^2 + [(n-1)a_1 - na_0]z + na_0$$

Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} < 1\ for\ i = 0,1,2,\ldots,n-2.$

Now $|Q(z)| \geq |a_{n-1}||z|^{n-1} - \{\ |a_{n-1} - 2a_{n-2}|\,|z|^{n-1} + |2a_{n-2} - 3a_{n-3}|\,|z|^{n-2} + \cdots + |(n-3)a_3 - (n-2)a_2||z|^3 + |(n-2)a_2 - (n-1)a_1||z|^2 + |(n-1)a_1 - na_0||z| + |na_0|\ \}$

$\geq |a_{n-1}||z|^{n-1}[\ |z| - \frac{1}{|a_{n-1}|}\{\ |a_{n-1} - 2a_{n-2}| + \frac{|2a_{n-2}-3a_{n-3}|}{|z|} + \cdots + \frac{|(n-3)a_3-(n-2)a_2|}{|z|^{n-4}} + \frac{|(n-2)a_2-(n-1)a_1|}{|z|^{n-3}} + \frac{|(n-1)a_1-na_0|}{|z|^{n-2}} + \frac{|na_0|}{|z|^{n-1}}\ \}\ ]$

$$\geq |a_{n-1}||z|^{n-1}[\ |z| - \frac{1}{|a_{n-1}|}\{\ |a_{n-1} - 2a_{n-2}| + |2a_{n-2} - 3a_{n-3}| + \cdots + |(n-3)a_3 - (n-2)a_2| + |(n-2)a_2 - (n-1)a_1| + |(n-1)a_1 - na_0| + |na_0|\ \}]$$

$$\geq |a_{n-1}||z|^{n-1}[\ |z| - \frac{1}{|a_{n-1}|}\{\ [a_{n-1} - 2a_{n-2}] + [2a_{n-2} - 3a_{n-3}] + \cdots + [(n-3)a_3 - (n-2)a_2] + [(n-2)a_2 - (n-1)a_1] + [(n-1)a_1 - na_0] + |na_0|\ \}]$$

$$\geq |a_{n-1}||z|^{n-1}[\ |z| - \frac{1}{|a_{n-1}|}\{\ a_{n-1} - na_0 + |na_0|\ \}]$$

> 0 if $|z| > \frac{1}{|a_{n-1}|}[\ a_{n-1} - na_0 + |na_0|\ ]$

This shows that if Q(z) > 0 provided $|z| > \frac{1}{|a_{n-1}|}[\ a_{n-1} - na_0 + |na_0|\ ]$

Hence all the zeros of Q(z) with |z|> 1 lie in $|z| \leq \frac{1}{|a_{n-1}|}[\ a_{n-1} - na_0 + |na_0|\ ]$.

But those zeros of Q(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of polar derivative $D_0P(z)$ are also the zeros of Q(z) lie in the circle defined by the above inequality and this completes the proof of the Theorem 1.

Proof of the Theorem 2.

Let P(z) = $a_n z^n + a_{n-1}z^{n-1} + \cdots + a_2 z^2 + a_1 z + a_0$ be a polynomial of degree n

Let denote $D_\alpha P(z) = nP(z) + (\alpha - z)P'(z)$ is polar derivative of P(z) with respect to real number α of degree n-1.

If $\alpha = 0$ then $D_0P(z) = nP(z) - zP'(z)$

$\Rightarrow D_0P(z) = a_{n-1}z^{n-1} + 2a_{n-2}z^{n-2} + 3a_{n-3}z^{n-3} + \cdots + (n-2)a_2z^2 + (n-1)a_1z + na_0$

Let us consider the polynomial Q(z) = (1 − z) $D_0P(z)$ so that

$$Q(z)=(1-z)(a_{n-1}z^{n-1} + 2a_{n-2}z^{n-2} + 3a_{n-3}z^{n-3} + \cdots + (n-2)a_2z^2 + (n-1)a_1z + na_0)$$

$$= -a_{n-1}z^n + (a_{n-1} - 2a_{n-2})z^{n-1} + (2a_{n-2} - 3a_{n-3})z^{n-2} + \cdots + [(n-3)a_3 - (n-2)a_2]z^3 + [(n-2)a_2 - (n-1)a_1]z^2 + [(n-1)a_1 - na_0]z + na_0$$

Also if $|z| > 1$ then $\frac{1}{|z|^{n-i}} < 1 \; for\; i = 0,1,2,\ldots,n-2.$

Now $|Q(z)| \geq |a_{n-1}||z|^{n-1} - \{ \; |a_{n-1} - 2a_{n-2}|\,|z|^{n-1} + |2a_{n-2} - 3a_{n-3}|\,|z|^{n-2} + \cdots + |(n-3)a_3 - (n-2)a_2||z|^3 + |(n-2)a_2 - (n-1)a_1||z|^2 + |(n-1)a_1 - na_0||z| + |na_0| \}$

$$\geq |a_{n-1}||z|^{n-1}[\, |z| - \frac{1}{|a_{n-1}|}\{ \,|a_{n-1} - 2a_{n-2}| + \frac{|2a_{n-2}-3a_{n-3}|}{|z|} + \cdots + \frac{|(n-3)a_3-(n-2)a_2|}{|z|^{n-4}} + \frac{|(n-2)a_2-(n-1)a_1|}{|z|^{n-3}} + \frac{|(n-1)a_1-na_0|}{|z|^{n-2}} + \frac{|na_0|}{|z|^{n-1}} \} \,]$$

$$\geq |a_{n-1}||z|^{n-1}[\, |z| - \frac{1}{|a_{n-1}|}\{ \,|a_{n-1} - 2a_{n-2}| + |2a_{n-2} - 3a_{n-3}| + \cdots + |(n-3)a_3 - (n-2)a_2| + |(n-2)a_2 - (n-1)a_1| + |(n-1)a_1 - na_0| + |na_0| \,\}]$$

$$\geq |a_{n-1}||z|^{n-1}[\, |z| - \frac{1}{|a_{n-1}|}\{ \,[2a_{n-2} - a_{n-1}] + [3a_{n-3} - 2a_{n-2}] + \cdots + [(n-2)a_2 - (n-3)a_3] + [(n-1)a_1 - (n-2)a_2] + [na_0 - (n-1)a_0] + |na_0| \,\}]$$

$$\geq |a_{n-1}||z|^{n-1}[\, |z| - \frac{1}{|a_{n-1}|}\{|na_0| + na_0 - \; a_{n-1} \,\}]$$

$$> 0 \text{ if } \; |z| > \frac{1}{|a_{n-1}|}[|na_0| + na_0 - \; a_{n-1}\,]$$

This shows that if Q(z) > 0 provided $|z| > \frac{1}{|a_{n-1}|}[|na_0| + na_0 - \; a_{n-1}\,]$

Hence all the zeros of Q(z) with |z|> 1 lie in $|z| \leq \frac{1}{|a_{n-1}|}[|na_0| + na_0 - \; a_{n-1}\,]$.

But those zeros of Q(z) whose modulus is less than or equal to 1 already satisfy the above inequality. Since all the zeros of polar derivative $D_0P(z)$ are also the zeros of Q(z) lie in the circle defined by the above inequality and this completes the proof of the Theorem 2..

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